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Stefan–Boltzmann Law for Black Bodies and Black Holes

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Since black holes radiate with a thermal spectrum and therefore possess a radiation pressure, Boltzmann's derivation of Stefan's Law can be applied to black holes. In order that the entropy be proportional to the surface area of the black hole, the pressure must be negative. If the second law is not to be violated, then the temperature must also be negative. This leads to a canonical formulation for fluctuations. A comparison with other approaches is given and doubts are raised concerning the validity of conventional black hole thermodynamics.

1. STEFAN'S LAW FOR BLACK BODIES

Boltzmann (1884) derived Stefan's law of thermal radiation from a global second law. Boltzmann's derivation of Stefan's law is based on an application of Carnot's cycle to the radiation enclosed within movable reflecting walls.

Let

$$e(T) = \int_0^{\infty} \rho(\nu, T)\, d\nu \tag{1}$$

be the total "aethereal" energy per unit volume in a cavity of volume V. Boltzmann wrote the second law in the form

$$\frac{\delta \mathcal{Q}}{T} = dS(eV, V) = \frac{1}{T}\{d(eV) + P\, dV\} \tag{2}$$

where $\delta\mathcal{Q}$ is the heat that must be added while the volume undergoes an alteration by the amount dV at constant temperature (Lord Rayleigh, 1902). Using the facts that for radiation, as well as aerial vibrations, the pressure

$$P = \tfrac{1}{3}e \tag{3}$$

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and that e is a function of the temperature T only, we have that the exactness condition is

$$\frac{\partial}{\partial V}\left(\frac{V}{T}\frac{de}{dT}\right)=\frac{\partial}{\partial T}\left(\frac{4}{3}\frac{e}{T}\right) \tag{4}$$

or

$$\frac{de}{dT}=\frac{4e}{T} \tag{5}$$

Integrating (5) gives Stefan's law

$$e=\sigma T^4 \tag{6}$$

where σ is $4/c$ times Stefan's constant.

An even neater proof of Stefan's law does not make any assumption about the energy density, which only invokes the constitutive relation $P=\frac{1}{3}e$, is based on the global form of the Gibbs-Duhem relation

$$d\left(\frac{P}{T}\right)+ed\left(\frac{1}{T}\right)=0 \tag{7}$$

Introducing the constitutive relation leads immediately to (5) and hence to Stefan's law.

2. LOCAL FORM OF STEFAN'S LAW

If the second law (2) is to hold globally, it must also hold *locally*; that is, for each frequency of the radiation field, namely

$$d\mathcal{S}_\nu(\rho_\nu V, V)=\frac{1}{T}\{d(\rho_\nu(T)V)+p_\nu(T)\,dV\} \tag{8}$$

However, whereas (2) is a purely thermodynamic relation, (8) lies midway between thermodynamics and statistical mechanics. In particular, there is no constitutive relation between the pressure per mode p_ν and the energy density per mode ρ_ν. The only relation between the two is in the form of the Gibbs-Duhem relation

$$d\left(\frac{p_\nu}{T}\right)+\rho_\nu\,d\left(\frac{1}{T}\right)=0 \tag{9}$$

whose integral is

$$\frac{p_\nu(T)}{T}=\int\frac{\rho_\nu}{T^2}\,dT \tag{10}$$

The Gibbs–Duhem relation (9) ensures the existence of the integral

$$\mathcal{S}_\nu(\rho_\nu V, V) = \frac{1}{T}(\rho_\nu + p_\nu)V \tag{11}$$

Notwithstanding the fact that both ρ_ν and p_ν are functions of both the frequency and temperature, the exactness conditions cannot be expressed in terms of the triplet ν, V, T, since ν is not a thermodynamic variable, either extensive or intensive. It is for this reason that we indexed the entropy in (11) by the frequency rather than considering it as a bona fide thermodynamic variable. This is supported by the fact that the Gibbs–Duhem relation (9) secures the exactness condition when V and T are taken as the independent variables.

In order to establish the validity of (11), it suffices to show that the sum over all modes coincides with the expression in (2). To accomplish this, an expression for the pressure is required. At this point, we must take recourse to a model as supplied by statistical mechanics, just as Planck unawaringly did in his treatment of the interaction of electromagnetic radiation and material "oscillators." Because the photon chemical potential vanishes for thermal radiation, the canonical and grand canonical ensembles coincide and we can use the work potential, per unit volume,

$$\begin{aligned} p_\nu(T) &= mkT \ln\left(\sum_{n=0}^{\infty} e^{-nh\nu/kT}\right) \\ &= -mkT \ln(1 - e^{-h\nu/kT}) \\ &\equiv kT \ln \xi_\nu(T) \end{aligned} \tag{12}$$

for the pressure per mode, where the density of states in the frequency interval $d\nu$ is $m = 8\pi\nu^2/c^3$, is Boltzmann's constant, and c is the velocity of light. In the last line of (12), we have expressed the pressure per mode in terms of the partition function per mode ξ_ν. With this expression for the pressure, the Gibbs–Duhem relation (9) gives the energy density per mode as

$$\rho_\nu(T) = \frac{mh\nu}{e^{h\nu/kT} - 1} = -\frac{\partial \ln \xi_\nu}{\partial \beta} \tag{13}$$

which is Planck's radiation law, which he discovered some 15 years after Boltzmann's proof of Stefan's law. We have introduced the inverse temperature as $k\beta = 1/T$. Finally, integrating (12) over all frequencies gives

$$P(T) = \int_0^\infty p_\nu(T)\, d\nu = \tfrac{1}{3}e(T) \tag{14}$$

This is a global constitutive relation of densities which are related through the Gibbs-Duhem equation (7).

The entropy per mode (11) can thus be expressed as the Legendre transform of the logarithm of the partition function per mode ξ_ν with respect to β as

$$\mathcal{S}_\nu(\rho_\nu V, V) = kV\left\{\ln \xi_\nu - \beta \frac{\partial \ln \xi_\nu}{\partial \beta}\right\} \tag{15}$$

We may thus define the free energy $\mathcal{F}_\nu$ of the mode of frequency ν as

$$\mathcal{F}_\nu(V, T) = mkTV \ln(1 - e^{-h\nu/kT}) \tag{16}$$

Denote by $\bar{n}_\nu$ the average number of particles per unit volume. Its relation to the energy density is

$$\rho_\nu(T) = \bar{n}_\nu(T)h\nu \tag{17}$$

Introducing (17) into (8) gives

$$d\mathcal{S}_\nu(\bar{n}_\nu V, V) = \frac{1}{T}\{(h\nu\bar{n}_\nu + p_\nu)\, dV + h\nu V\, d\bar{n}_\nu\} \tag{18}$$

Analogous to the global Gibbs-Duhem relation (7) there is the local Gibbs-Duhem relation per mode (9), where the energy density is given by (17). Evaluating (9) with the aid of (12) yields

$$\bar{n}_\nu(T) = \frac{m}{e^{h\nu/kT} - 1} \tag{19}$$

In the same way that the global Gibbs-Duhem relation (7) gives Stefan's law (6), the Gibbs-Duhem relation per mode (9), gives the Bose distribution (19) when the pressure per mode is given by (12). We may consider (19) as the condition for a dynamical equilibrium between the osmotic pressure force, proportional to $kd \ln \xi_\nu$, and the thermal force, proportional to $d(1/T)$.

Because photons do not interact with one another, they cannot achieve thermal equilibrium by themselves. Thermal equilibrium is achieved when the chemical potential of the photons is equal to the vanishing chemical potential of what they interact with, for example, the chemical potential of electron-hole pairs in a light-emitting diode (Würfel, 1982). An applied voltage to a p-n junction brings the system out of equilibrium with the consequence that the photon chemical potential is nonzero. For nonthermal radiation, the entropy per mode (11) must be generalized to

$$\mathcal{S}_\nu(\rho_\nu V, V, \bar{n}_\nu V) = \frac{1}{T}(\rho_\nu + p_\nu - \mu\bar{n}_\nu)V \tag{20}$$

where μ is the photon chemical potential and

$$p_\nu(T) = -mkT \ln(1 - e^{-(h\nu-\mu)/kT}) \tag{21}$$

is the photon pressure in mode ν.

On account of the first-order homogeneous property of the entropy, (20) can be scaled according to $\mathscr{S}_\nu(\rho_\nu V, V, \bar{N}_\nu) = V\mathscr{S}_\nu(\rho_\nu, 1, \bar{n}_\nu) \equiv s_\nu(\rho_\nu, \bar{n}_\nu)$, where $\bar{n}_\nu = \bar{N}_\nu / V$. The total differential of s_ν is

$$ds_\nu(\rho_\nu, \bar{n}_\nu) = \left(\frac{\partial s_\nu}{\partial \bar{n}_\nu}\right)_{v,\rho_\nu} d\bar{n}_\nu + \left(\frac{\partial s_\nu}{\partial \rho_\nu}\right)_{v,\bar{n}_\nu} d\rho_\nu$$

but because ρ_ν and $\bar{n}_\nu$ are not independent variables, since they are related linearly by (17), the entropy differential can be expressed as $ds_\nu = (\partial s/\partial \bar{n}_\nu)_V \, d\bar{n}_\nu$, where

$$\begin{aligned}\left(\frac{\partial s_\nu}{\partial \bar{n}_\nu}\right)_V &= \left(\frac{\partial s_\nu}{\partial \bar{n}_\nu}\right)_{V,\rho_\nu} + \left(\frac{\partial s_\nu}{\partial \rho_\nu}\right)_{V,\bar{n}_\nu} \frac{\partial \rho_\nu}{\partial \bar{n}_\nu} \\ &= -\frac{\mu}{T} + \frac{h\nu}{T}\end{aligned} \tag{22}$$

3. ENTROPY AND PROBABILITY

The general relationship between the entropy and probability is (Lavenda, 1988)

$$k \ln f(n; \bar{n}_\nu) = -(n - \bar{n}_\nu)\left(\frac{\partial s_\nu}{\partial \bar{n}_\nu}\right)_v - s_\nu(\bar{n}_\nu) + s(n) \tag{23}$$

where $f(n; \bar{n}_\nu)$ is the law of error for which the mean of the distribution is the most probable value. Provided m and n are sufficiently large to warrant Stirling's approximation, the "fluctuation" entropy density $s(n)$ is the same function of the random quantity n that the entropy density per mode $s_\nu(\bar{n}_\nu)$ is of the mean value $\bar{n}_\nu$ (Lavenda and Dunning-Davies, 1990). In contrast to (23), Boltzmann's principle is

$$s(n) = k \ln \Omega(n) \tag{24}$$

where $\Omega(n)$ is the number of ways that n *indistinguishable* particles can be put into m cells in the case of Bose statistics. Ω can therefore not be considered as a probability at all, but rather a large positive integer. The adjective "thermodynamic" has been reserved to qualify such a "probability." Now the connection with entropy is achieved by maximizing Boltzmann's principle (24),

$$s_\nu(\bar{n}_\nu) = k \ln \Omega_{\max} \tag{25}$$

where it turns out that the maximum of $\Omega(n)$ occurs at $n=\bar{n}_\nu$ so that $\Omega_{\max}=\Omega(\bar{n}_\nu)$.

With the negative binomial distribution as the law of error, the entropy density per mode is given by

$$s_\nu(\bar{n}_\nu)=k\left\{\bar{n}_\nu \ln\left(\frac{\bar{n}_\nu+m}{\bar{n}_\nu}\right)+m\ln\left(\frac{\bar{n}_\nu+m}{m}\right)\right\} \tag{26}$$

Taking the derivative of (26) and setting it equal to (22) gives

$$\bar{n}_\nu(T)=\frac{m}{e^{(h\nu-\mu)/kT}-1} \tag{27}$$

which is the generalization of the Bose distribution (19) to nonzero chemical potential.

4. STEFAN'S LAW FOR BLACK HOLES

The derivation of Stefan's law for black holes leads to an understanding of the nature of the equilibrium that is established between the black hole and surrounding black-body radiation which is necessary in order to be able to define a temperature.

Let the black hole be enclosed in a volume $V>V_s$, where V_s is the Schwarzschild volume, which contains a certain amount of radiant energy. The change in the entropy of a black hole caused by the absorption of radiant energy is

$$d\mathcal{S}=\frac{1}{T}\{d(eV)+P\,dV\} \tag{28}$$

The condition that (28) be a perfect differential is

$$\frac{\partial}{\partial V}\left(\frac{V}{T}\frac{de}{dT}\right)=\frac{\partial}{\partial T}\left(\frac{e+P}{T}\right) \tag{29}$$

The only relation that will give an inverse dependence of the energy density on the temperature which the conventional interpretation of black-hole thermodynamics requires is

$$e=-2P \tag{30}$$

since inserting (30) into (29) gives

$$\frac{de}{dT}=-\frac{e}{T} \tag{31}$$

Integrating (31) gives Stefan's law for black holes as

$$e = \frac{\alpha}{T} \tag{32}$$

or $e \cdot T = \text{const}$, and α is some constant to be determined.

It is clear from the constitutive relation (30) that the pressure must be negative, indicating that the object under consideration is contracting spontaneously so as to increase its entropy. Spontaneous contraction of an imploding star would lead to the formation of a new surface, leading to a metastable state which in the case of a black hole would be its horizon. Negative pressures in a cosmological model are not unknown (McGrea, 1951). General relativity does not restrict the pressure to positive values (Hawking and Ellis, 1973, p. 90) and (30) does not violate the dominant energy condition, which states that the velocity of energy flow is always less than the speed of light. For an energy density which scales as the inverse square of the radius of curvature of the universe, the adiabatic condition yields a negative pressure (Görnitz, 1988). A negative pressure has even been implicated in the inflationary scenario of the hot big-bang cosmology (Guth, 1981). However, if the second law is not to be invalidated, a negative pressure cannot be "the driving force behind exponential expansion" (Guth, 1981) because $P<0$ implies $(\partial\mathcal{S}/\partial V)_E<0$ and the system would spontaneously contract in order that its entropy increases (Landau and Lifshitz, 1969, p. 42).

In the process of the collapse of an object to form a black hole, energy is radiated away in the form of gravity waves. By applying the techniques of second quantization, Hawking (1974) has shown that the radiation settles down, as the collapse winds up, to the thermal radiation that a black body would have. Thus, E is the total mass-energy of a black hole, which includes the energy of black-body radiation and gravitational radiation, since the black hole can still produce gravity waves when it swallows other matter, which should be most intense when two equal-mass black holes collide. Electromagnetic radiation produces a radiation pressure P that is equal to one-third of its energy density. We shall assume that the pressure of matter is only a minute fraction of its mass-energy density, so that it can be neglected for a black hole as well as for matter under ordinary circumstances (Tolman, 1934, p. 273). Although this is a reasonable assumption, it is not clear from Hawking's conjecture why gravitational radiation, with a negligible radiation pressure, should settle down, in the final phase of collapse, to radiation that has a thermal spectrum with a definitely nonnegligible radiation pressure.

It has long been known (Bartoli, 1884) that a radiation pressure is needed in order not to violate the second law when radiant energy is transported from a cold body to a hot one by means of a moving mirror. A pressure must be exerted on the mirror by the light. That heat cannot be transported from a cold to a hot body without any other work being done on the system means that the constant α appearing in Stefan's law (32) must be negative (Lavenda and Dunning-Davies, 1988). The greater the radiated energy, the greater must be the temperature and this can only be true if the temperature is negative. The constant α must have units of energy squared divided by Boltzmann's constant. Setting $\alpha = -\sigma_e^2/k < 0$ allows us to write Stefan's law (32) as

$$e = -\frac{\sigma_e^2}{kT} \tag{33}$$

The Gibbs–Duhem relation (7) ensures that the integral of (28) exists and is given by

$$\mathcal{S} = \frac{V}{T}\{e + P\} \tag{34}$$

Introducing (30) and (33) for the black-hole temperature into the black-hole entropy (34) gives

$$\mathcal{S} = -k\frac{e^2}{2\sigma_e^2}V = -k\frac{E^2}{2V\sigma_e^2} \tag{35}$$

The negative sign in (35) should cause no concern when it is recalled that only entropy differences have any physical meaning in thermodynamics. And although the black-hole entropy is a quadratic function of the radiation energy, it is nonetheless an *extensive* quantity. This expression (35) bears only a formal resemblance to the expression proposed by Bekenstein (1972) in which he set the black-hole entropy proportional to the surface area. For a Schwarzschild hole, the surface area is proportional to the square of the mass. Bekenstein's expression is based on an analogy between Hawking's area theorem, stating that the surface area of the event horizon cannot decrease in any dynamical process, and the second law. Furthermore, the temperature defined thermodynamically as

$$\left(\frac{\partial \mathcal{S}}{\partial E}\right)_V = \frac{1}{T} = -k\frac{E}{V\sigma_e^2} \tag{36}$$

is an intensive thermodynamic variable.

In the conventional formulation of black-hole thermodynamics, it is concluded that black holes grow hotter when they radiate heat. It has long been known that systems in contact with a heat reservoir cannot have negative heat capacities (Lynden-Bell and Wood, 1967). It is precisely because of the negative sign in (35) that the heat capacity at constant volume C_v appearing in the expression

$$\left(\frac{\partial^2 \mathscr{S}}{\partial E^2}\right)_V = -(T^2 C_V)^{-1} = -\frac{k}{V\sigma_e^2} < 0 \tag{37}$$

is positive. But, in the canonical ensemble $kT^2 C_V$ is the mean square fluctuation in energy and this identifies σ_e^2 as the second moment of the distribution. Finally, eliminating the energy density between (30) and (33) gives

$$P = \frac{\sigma_e^2}{2kT} \tag{38}$$

which is the equation of state for a black hole.

We now compare the sum of the individual radiation entropies of the two black holes (labeled by indices 1 and 2) and each of which is located somewhere in a box of volume V, with the entropy of a black hole which is formed by the collision and coalescence of two black holes (characterized by no indices) in a box of twice the volume. The box is assumed large enough so that its volume is greater than the Schwarzschild volume of the combined black hole. The Legendre transform of the entropy with respect to the energy is

$$\mathscr{S}_i - \left(\frac{\partial \mathscr{S}_i}{\partial E_i}\right)_V E_i = k \ln \mathscr{Z}_i(\beta_i), \qquad i = 1, 2 \tag{39}$$

where

$$\ln \mathscr{Z}_i = \tfrac{1}{2}\beta_i^2 \sigma_e^2 V \tag{40}$$

is the logarithm of the partition function. From (40) it is clear that the radiation pressure of the individual black holes will be the same if their temperatures are the same, since we have assumed that they are enclosed in equal volumes. In the canonical ensemble it is more natural to consider the entropy

$$\mathscr{S}_i = k\{\beta_i E_i + \ln \mathscr{Z}_i\} \tag{41}$$

as a function of $\beta_i = k^{-1}(\partial \mathscr{S}_i / \partial E_i)_V$ rather than the energy E_i.

When two black holes collide they can emit gravity waves. The conservation of energy requires

$$E = E_{bh} + E_g = E_1 + E_2 \tag{42}$$

where E is the sum of mass-radiation energy of the final black hole E_{bh} and the gravitational radiation energy, E_g. Since we have assumed that the radiation pressure of gravity waves is negligible with respect to the radiation pressure of thermal radiation, which is one-third its energy density, the radiation pressure of the combined black hole will have the same pressure as the individual black holes given in (38). Since the individual black holes do not emit gravity waves, the radiation pressure of gravitational radiation is negligible in comparison to the thermal radiation pressure. Hence, the contribution to the entropy of the combined black hole due to gravitational radiation is approximately $\mathscr{S}_g \approx k\beta E_g$. Consequently,

$$\begin{aligned}\mathscr{S}_1(\beta)+\mathscr{S}_2(\beta) &= k\{\beta E_1+\ln \mathscr{Z}_1+\beta E_2+\ln \mathscr{Z}_2\}\\ &= k\{\beta(E_{bh}+E_g)+\ln \mathscr{Z}\}\\ &= \mathscr{S}_{bh}(\beta)+\mathscr{S}_g(\beta)=\mathscr{S}(\beta)\end{aligned} \tag{43}$$

where the entropy of the combined black hole $\mathscr{S}$ has been decomposed into a black hole entropy $\mathscr{S}_{bh}$ that it would have in the absence of gravitational waves and a contribution $\mathscr{S}_g$ due to gravitational radiation that has a negligibly small radiation pressure.

Since $\mathscr{S}_i$ achieve their minima at β_i, $\mathscr{S}_i(\beta)\geq \mathscr{S}_i(\beta_i)$ we have

$$\mathscr{S}(\beta)\geq \mathscr{S}_1(\beta_1)+\mathscr{S}_2(\beta_2) \tag{44}$$

when the two black holes are not initially at the same temperature. Note that we cannot say anything about the black hole entropy $\mathscr{S}_{bh}$ in relation to the individual entropies when there is significant loss of energy through gravitational radiation. If the gravitational radiation pressure were of the same order of magnitude as the mass-energy density, no inequality could be derived between the final and initial entropies.

Barring this case, the second law (44) may be seen to imply

$$E_{bh}^2 \leq E^2 \leq 2(E_1^2+E_2^2) \tag{45}$$

on account of the negative quadratic form of the entropy of a black hole (35) and the fact that the final volume is twice as great as the individual volumes enclosing each of the black holes. If the two black holes have the same mass-energy, E_1 say, then the maximum gravitational energy that can be released upon collision is $2E_1 - E_{bh}$. Since radiant energy must be included in the conservation of energy, it is doubtful whether the area theorem involving only the rest energy of a Schwarzschild black hole has any connection with the second law.

In fact, the entropy can provide no information on how the areas of black holes behave when they collide and coalesce to form a single black hole. If the second law could be used to establish the area theorem, then it would be necessary to show that the entropy of a black hole formed from the coalescence of two holes is the sum of the entropies when the individual black holes initially have the same surface gravity, provided, again, that the gravitational radiation pressure can be neglected. This would be analogous to the additivity of the entropy (43) when the individual systems are initially at the same temperature.

Radiation processes are, in essence, random phenomena. Let ε denote the random energy density whose average value is e. On one hand, we have the expression for the partition function (40), while on the other hand, the partition function is the Laplace transform of the "structure" function Ω,

$$\mathscr{Z}(\beta) = \int e^{-\beta\varepsilon}\Omega(\varepsilon)\, d\varepsilon \tag{46}$$

where the lower limit of integration can be extended to $-\infty$ (Blanc-Lapierre and Tortat, 1956). These two facts imply that the "structure" function is given by the normal density,

$$\Omega(\varepsilon) = (2\pi\sigma_e^2/V)^{-1/2}\exp(-\varepsilon^2 V/2\sigma_e^2) \tag{47}$$

Then, in view of the error law (23), the probability density function for the radiant energy of a black hole is given by

$$f(\varepsilon;\, e) = (2\pi\sigma_e^2/V)^{-1/2}\exp[-(\varepsilon - e)^2 V/2\sigma_e^2] \tag{48}$$

The only information that went into the derivation Gauss' law of error (48) is that the entropy has the quadratic form (35). This, in turn, followed from expression (30) relating the mass-energy density of a black hole to the (negative) radiation pressure and the form of Stefan's law (33). Therefore, the probability density for the radiant energy of a black hole will have the Gaussian form (48).

The Gaussian law of error (48) is ordinarily the small-fluctuation limit of each of the two forms of statistics. However, we have derived it without any assumption regarding the size of the fluctuations and, consequently, it is to be considered as an *exact* relation. It therefore follows that if it is possible to express the average energy density e in terms of the mean number of quanta $\bar{n}_\nu$ as

$$e(T) = h\int_0^\infty \nu\bar{n}_\nu(T)\, d\nu \tag{49}$$

then the $\bar{n}_\nu$ will not have the form of either of the two known forms of statistics.

5. DISCUSSION

The scope of the foregoing analysis has been to draw some thermodynamic conclusions from Hawking's theorem that "black-hole surface area cannot decrease" without violating the second law of thermodynamics, which states that "heat cannot spontaneously flow from a cold to a hot body without any other work being done on the system." It has often been claimed that the area theorem, being classical in nature, can be violated by quantum processes and, in particular, the thermal radiation discovered by Hawking. One then turns to a generalized second law in which the increase in the exterior entropy due to the radiation offsets the entropy decrease of the black hole (Bekenstein, 1975). This appears a little odd insofar as Planck never found it necessary to invoke the entropy of the universe in his application of the second law to black-body radiation. If the entropy is to have a thermodynamic meaning, it cannot be determined solely from the mass, charge, and angular momentum of an object. The notion of heat transfer has to appear somewhere in the derivation and, in this sense, conventional black-hole thermodynamics shares a common feature with the introduction of entropy in general relativity for nonadiabatic models.

If a black hole does radiate with a thermal spectrum, it must necessarily satisfy the black-body relation (3) and not (30). Yet, it is precisely that relation which is required to give an inverse relation between the energy density and temperature in order that the entropy be proportional to the surface area of a black hole.

Cracks in the argument relating the entropy of a Schwarzschild black hole to the square of its mass have been found. Hawking (1976) has noticed that "although the canonical ensemble does not work for black holes, one can still employ a microcanonical ensemble of a large number of similar insulated systems each with a given fixed energy." The reason for this is that having taken the negative of (35) as the expression for the entropy, the structure function, instead of being an exponentially decreasing function of the square of the energy as in (47), is an exponentially increasing function of the square of the energy and as such it cannot be overpowered by the thermal factor $\exp(-\beta\varepsilon)$ in the expression for the partition function (46) in order to make the latter a finite quantity. Yet, a microcanonical ensemble must be of finite, macroscopic extension and there is nothing to prohibit us from focusing on a small, but macroscopic, part of it that is in thermal contact with the rest. This is the usual way that a canonical ensemble is defined. That this cannot be accomplished puts grave doubts on the expression for the black-hole entropy and its relation to a "thermodynamic" probability (Hawking, 1976). In thermodynamics such systems are outlawed precisely by the fact that the heat capacity must be positive.

We have retained the conventional black-hole thermodynamic functional form of the expression for the entropy and temperature while renouncing a negative heat capacity. The price we paid is extremely high, for it necessitates the introduction of both a negative radiation pressure and temperature. If such systems exist, they can only be metastable. The question then arises of how such systems can be placed in thermal contact with other systems in order to achieve an overall stable system. The problem is aggravated even more if one wants to hold on to a positive temperature while relinquishing a positive heat capacity for a black hole. Hawking (1976) has considered the thermal equilibrium stability requirement for a black hole to be in equilibrium with black-body radiation in a volume greater than the Schwarzschild volume. Using the method of a composite system, he derived the equality of the temperatures from the vanishing of the first variation of the entropy of the composite system, while he obtained an inequality for the temperature to satisfy from the criterion that the second variation be negative. A thermodynamic stability criterion which yields a critical value of the temperature is completely foreign to Carathéodory's use of composite systems in order to show that the entropy of a less constrained system should not be inferior to the sum of the entropies of the individual subsystems. But those subsystems must be of the same type, for otherwise the temperature would have to obey an inequality for thermodynamic stability. All that can be deduced when two nonidentical systems are placed in thermal contact is that the temperature of the two systems must be equal and the second variations of their entropies must each be less than zero in order for the "composite" system to be thermodynamically stable. But this is impossible for a black hole, which possesses a negative heat capacity.

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